

Calmness of partially perturbed linear systems with an application to the central path*

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Abstract

In this paper we develop point-based formulas for the calmness modulus of the feasible set mapping in the context of linear inequality systems with a fixed abstract constraint and (partially) perturbed linear constraints. The case of *totally* perturbed linear systems was previously analyzed in [9, Section 5]. We point out that the presence of such an abstract constraint yields the current paper to appeal to a notable different methodology with respect to previous works on the calmness modulus in linear programming. The interest of this model comes from the fact that partially perturbed systems naturally appear in many applications. As an illustration, the paper includes an example related to the classical central path construction. In this example we consider a certain feasible set mapping whose calmness modulus provides a measure of the convergence of the central path. Finally, we underline the fact that the expression for the calmness modulus obtained in this paper is (conceptually) implementable as far as it only involves the nominal data.

Key words. Calmness, local error bounds, linear programming, feasible set mapping, interior point methods.

Mathematics Subject Classification: 90C31, 49J53, 90C05, 90C51.

*Dedicated to J. Outrata on his 70th Birthday

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1 Introduction

In this paper we consider a parametrized linear inequality system written in the form

$$\{x \in C; a'_t x \leq b_t, t \in I\}, \quad (1)$$

where $x \in \mathbb{R}^n$ is the vector of *decision variables*, $\emptyset \neq C \subset \mathbb{R}^n$ is a closed convex set, I is a nonempty finite set and

$$(a, b) := (a_t, b_t)_{t \in I} \in (\mathbb{R}^n \times \mathbb{R})^I$$

is the *parameter* to be perturbed around a nominal value $(\bar{a}, \bar{b}) := (\bar{a}_t, \bar{b}_t)_{t \in I}$. The case of ordinary linear systems, when $C = \mathbb{R}^n$, is included in the current setting (' \subset ' is understood as the nonstrict inclusion). We also assume that $\bar{a}_t \neq 0_n$ for all $t \in I$. Along the paper, vectors in \mathbb{R}^n are considered as column vectors and the prime represents the transpose (so, $a'_t x$ is the usual scalar product of a_t and x).

In this context, we consider the *feasible set mapping*, $\mathcal{F} : (\mathbb{R}^n \times \mathbb{R})^I \rightrightarrows \mathbb{R}^n$, given by

$$\mathcal{F}(a, b) := \{x \in C \mid a'_t x \leq b_t, t \in I\}, \quad (2)$$

for all $(a, b) \in (\mathbb{R}^n \times \mathbb{R})^I$. Observe that C remains fixed in our analysis, i.e. it is not subject to perturbations; recall that ' $x \in C$ ' is called an *abstract constraint*.

It is well-known that any closed convex set may be written as the solution set of a linear semi-infinite inequality system (see [15] for a comprehensive study of such systems), so \mathcal{F} can be seen as the feasible set mapping associated with a *partially* perturbed linear system. Formally, we can write (alternatively to (2))

$$\mathcal{F}(a, b) := \{x \in \mathbb{R}^n \mid c'_s x \leq d_s, s \in U; a'_t x \leq b_t, t \in I\},$$

for some (possibly infinite) index set U , disjoint with I , which is used for describing our abstract constraint as an 'unperturbed' system of linear inequalities.

The main goal of this work is to compute the *calmness modulus* of \mathcal{F} at the nominal element $((\bar{a}, \bar{b}), \bar{x}) \in \text{gph} \mathcal{F}$ (the graph of \mathcal{F} ; i.e., $\bar{x} \in \mathcal{F}(\bar{a}, \bar{b})$). As an immediate antecedent, the reader is addressed to [9, Section 5] where a formula for the calmness modulus of the \mathcal{F} in the setting of *totally* perturbed linear systems (with $U = \emptyset$) is provided. To this respect, the current work generalizes to systems (1) the results of [9, Section 5].

At this moment, we underline the fact that the inclusion of such an abstract constraint in the current paper entails notable differences with respect to the referred previous works; in fact, the technical tools from convex analysis (specifically, from subdifferential calculus) used here and, more generally, the methodology followed in this paper are completely different from previous works, where more direct algebraic arguments of finite Euclidean spaces are applied.

Roughly speaking, the calmness modulus of \mathcal{F} at $((\bar{a}, \bar{b}), \bar{x})$ provides the ratio of local enlargement of the feasible sets (around the nominal $\bar{x} \in \mathcal{F}(\bar{a}, \bar{b})$) with respect to perturbations of the data, and the calmness property prevents abrupt local enlargements (with a nonlinear rate) with respect to the parameter perturbations. See Subsection 2.2 for the formal definitions. The calmness property plays a key role in many issues of mathematical programming like optimality conditions, error bounds or stability of solutions, among others; the reader is addressed to the monographs [11, 24, 33, 38] for a comprehensive study of this and other variational properties. In the last decades there has been a growing interest in criteria for calmness and related concepts, as *local error bounds* (see, for instance, [4, 12, 18, 19, 22, 25, 27, 31]), or *Hoffman constants* (see, e.g., [26, 28, 29]). See [20, 21], in the context of single-valued maps, for the study of this property in connection with necessary optimality conditions.

As an illustration, in Section 4, we study a certain feasible set mapping appearing in connection with the well-known central path construction; see, e.g., the classical works of [30] and [32], and references therein (see also [6, Chapter 9] for some details and additional references). Specifically, we consider a linear programming problem in standard form, (P) , and for each $\mu > 0$ the associated logarithmic barrier problem, (P_μ) ,

$$\begin{array}{ll} (P) \quad \min & c'x \\ \text{s.t.} & Ax = b, \\ & x \geq 0_n, \end{array} \quad \begin{array}{ll} (P_\mu) \quad \min & c'x - \mu \sum_{i=1}^n \log x_i \\ \text{s.t.} & Ax = b, \\ & x > 0_n, \end{array} \quad (3)$$

where $x \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, 0_n is the null vector in \mathbb{R}^n , A is a $m \times n$ real matrix and $b \in \mathbb{R}^m$. Under the assumptions introduced in Section 4, (P_μ) has a unique optimal solution, $x(\mu)$, and it can be characterized as a part of the solution of the non-linear system which arises from the Karush-Kuhn-Tucker (KKT, for short) optimality conditions. If we denote by $(x(\mu), y(\mu), z(\mu))$ the complete solution of this KKT system (including dual variables; see Section 4 for details), the path $\{(x(\mu), y(\mu), z(\mu)); \mu > 0\}$ is usually referred to as the *central path* associated with LP problem (P) ; see the complete

description in [32, Section 2]. The example of Section 4 provides explicit expressions for constants $\kappa \geq 0$ satisfying

$$d((x(\mu), y(\mu), z(\mu)), \Lambda) \leq \kappa\mu \text{ for } \mu > 0 \text{ sufficiently small,} \quad (4)$$

where $d((x(\mu), y(\mu), z(\mu)), \Lambda)$ is the distance from $(x(\mu), y(\mu), z(\mu))$ to the subset $\Lambda \subset \mathbb{R}^n \times (\mathbb{R}^m \times \mathbb{R}^n)$ of all pairs of primal-dual optimal solutions of (P) . At this moment, we advance that these constants κ are given in terms of the calmness modulus of a certain feasible set mapping which is defined in (25) and of the existing limit point

$$(x^0, y^0, z^0) := \lim_{\mu \rightarrow 0} (x(\mu), y(\mu), z(\mu));$$

see [3] and [10] for the analysis of this limit point (indeed, for different choices of penalty and barrier functions apart from the logarithmic barrier considered in (3)).

Let us comment here that the formula for κ only depends on c , A , b , and on (x^0, y^0, z^0) , and so, it is conceptually implementable since it involves only fixed elements. See Section 5 (of conclusions) for details about the relationship between (4) and classical results as [17, Theorem 2.1 and Corollary 2.2] on the rate of convergence of $x(\mu)$ and $z(\mu)$ to x^0 and z^0 in terms of their derivatives.

Now we describe the outline of the paper. Section 2 provides the notation and preliminary results. Section 3 contains the main results (and the most technical difficulties) in this paper; they are focussed on the calmness modulus of the feasible mapping \mathcal{F} , introduced in (2), at the nominal element $((\bar{a}, \bar{b}), \bar{x}) \in \text{gph}\mathcal{F}$. This section is divided into three subsections. Subsection 3.1 is concerned with the calmness modulus of the mapping denoted by $\mathcal{F}_{\bar{a}}$, corresponding to the case where perturbations fall exclusively on the right-hand side terms b_t , $t \in I$, while the left-hand side members of the constraints remain fixed at \bar{a} (this motivates the notation $\mathcal{F}_{\bar{a}}$). Subsection 3.2 deals with the case when C is a polyhedral set, expressed explicitly by means of linear inequalities. Subsection 3.3 is addressed to calculate the calmness modulus of \mathcal{F} for two-sided perturbed inequality systems. Section 4 applies the results derived in the previous section for obtaining an explicit expression for constant κ in (4). We finish the paper with a brief section of conclusions.

2 Notation and preliminaries

This section gathers some necessary notation and results used in the paper. For the sake of clarity, the section is divided into two subsections containing, respectively, some basic tools of convex analysis in \mathbb{R}^n and some preliminaries about calmness and local error bounds. Throughout the paper, the space of variables, \mathbb{R}^n , is equipped with an arbitrary norm, $\|\cdot\|$, whose corresponding *dual norm* is given by $\|u\|_* = \max_{\|x\| \leq 1} |u'x|$ and d_* refers to the distance associated with the dual norm $\|\cdot\|_*$.

2.1 Basic tools of convex analysis

Given $X \subset \mathbb{R}^n$, we denote by $\text{conv}X$, $\text{cone}X$ and X^- the *convex hull*, the *conical convex hull* and the (*negative*) *dual cone* of X , respectively. Remember that

$$X^- = \{z \in \mathbb{R}^n \mid x'z \leq 0 \text{ for all } x \in X\}.$$

It is assumed that $\text{cone}X$ always contains the zero-vector, in particular $\text{cone}(\emptyset) = \{0_n\}$.

If Y is another set in \mathbb{R}^n we define

$$X + Y := \{x + y \mid x \in X, y \in Y\},$$

with the conventions

$$X + \emptyset = \emptyset + Y = \emptyset.$$

If $\Lambda \subset \mathbb{R}$, we also define

$$\Lambda X := \{\lambda x \mid \lambda \in \Lambda, x \in X\},$$

and $\Lambda \emptyset = \emptyset X = \emptyset$.

Along this paper we also use the usual *normal cone* of X at x :

$$N_X(x) := \begin{cases} \{z \in \mathbb{R}^n \mid (y - x)'z \leq 0 \text{ for all } y \in X\} = (X - x)^-, & \text{if } x \in X, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In the topological side, $\text{int}X$, $\text{ri}X$, $\text{cl}X$ and $\text{bd}X$ stand, respectively, for the *interior*, the *relative interior*, the *closure* and the *boundary* of X . Obviously, if $x \in \text{int}X$, then $N_X(x) = \{0_n\}$.

If X is convex, Farkas lemma provides the following relationship:

$$N_X(x)^- = (X - x)^{- -} = \text{cl}\mathbb{R}_+(X - x),$$

which is nothing else but the *tangent cone* to X at x , denoted by $T_X(x)$; i.e.

$$N_X(x)^- = T_X(x).$$

The following lemma establishes an elementary property which is used later in the paper.

Lemma 1 *Assume that X is a convex set in \mathbb{R}^n and $\bar{x} \in X$. If $d \neq 0_n$ and $\lambda_d > 0$ are such that $\bar{x} + \lambda d \in X$ whenever $0 < \lambda < \lambda_d$, then*

$$N_X(\bar{x} + \lambda d) = N_X(\bar{x}) \cap \{d\}^\perp, \text{ for all } 0 < \lambda < \lambda_d.$$

Proof. Let us see the inclusion ' \subset '. Fix any $0 < \lambda < \lambda_d$ and take $u \in N_X(\bar{x} + \lambda d)$; let us show first that $u \in \{d\}^\perp$. On the one hand, if we take any $\lambda_1 > 0$ such that $\lambda < \lambda_1 < \lambda_d$, since $\bar{x} + \lambda_1 d \in X$, we have

$$u'((\bar{x} + \lambda_1 d) - (\bar{x} + \lambda d)) = (\lambda_1 - \lambda) u'd \leq 0,$$

which entails $u'd \leq 0$. On the other hand,

$$u'(\bar{x} - (\bar{x} + \lambda d)) = -\lambda u'd \leq 0;$$

so, $u'd = 0$.

Now, one easily see that $u \in N_X(\bar{x})$. Indeed, for any $x \in X$, we have

$$u'(x - \bar{x}) = u'(x - (\bar{x} + \lambda d)) \leq 0.$$

Now let us establish the inclusion ' \supset '. Again, take any $0 < \lambda < \lambda_d$. We have that,

$$\begin{aligned} N_X(\bar{x}) \cap \{d\}^\perp &= \{u \in \mathbb{R}^n \mid u'(x - \bar{x}) \leq 0 \text{ for all } x \in X, u'd = 0\} \\ &= \{u \in \mathbb{R}^n \mid u'(x - (\bar{x} + \lambda d)) \leq 0 \text{ for all } x \in X, u'd = 0\} \\ &\subset N_X(\bar{x} + \lambda d). \end{aligned}$$

■

We say that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *proper* if its (*effective*) *domain*, $\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$, is nonempty. We say that f is *convex* (*lower semicontinuous* or *lsc*, for short, respectively) if its *epigraph*, $\text{epi } f := \{(x, \lambda) \in \mathbb{R}^{n+1} \mid f(x) \leq \lambda\}$, is convex (closed, respectively).

The *Fenchel subdifferential* of f at a point $x \in \text{dom } f$ is the closed convex set

$$\partial f(x) := \{z \in \mathbb{R}^n \mid f(y) \geq f(x) + z'(y - x) \text{ for all } y \in \mathbb{R}^n\}.$$

If $x \notin \text{dom } f$, then we set $\partial f(x) = \emptyset$.

Given two proper convex functions $f, g : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, the classical Rockafellar qualification condition

$$\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset \quad (5)$$

ensures that (see [37, Theorem 23.8]):

$$\partial(f + g)(x) = \partial f(x) + \partial g(x). \quad (6)$$

The *support* and the *indicator* functions of $X \subset \mathbb{R}^n$ are, respectively, defined as

$$\sigma_X(y) := \sup\{x'y \mid x \in X\}, \text{ for } y \in \mathbb{R}^n,$$

assuming $\sigma_\emptyset \equiv -\infty$, and

$$I_X(x) := \begin{cases} 0, & \text{if } x \in X, \\ +\infty, & \text{if } x \in \mathbb{R}^n \setminus X. \end{cases}$$

The function σ_X is convex (sublinear, indeed) and lsc, whereas I_X is convex and lsc if and only if X is a closed convex set.

Given a proper convex function $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$, if f is continuous at $x \in \text{dom } f$, the directional derivative function $f'(x; \cdot)$ is continuous and

$$f'(x; d) = \sigma_{\partial f(x)}(d) = \max\{u'd \mid u \in \partial f(x)\}, \text{ for } d \in \mathbb{R}^n. \quad (7)$$

It is well-known that if $X \subset \mathbb{R}^n$ is a convex set and $x \in X$, then

$$\partial I_X(x) = N_X(x). \quad (8)$$

If X is a polyhedral set, with explicit representation

$$X = \{x \in \mathbb{R}^n \mid c'_s x \leq d_s, \ s \in S\},$$

where S is a non-empty finite set, for $x \in X$ one has

$$N_X(x) = \text{cone}\{c_s \mid s \in S(x)\}, \quad (9)$$

where

$$S(x) := \{s \in S \mid c'_s x - d_s = 0\}.$$

2.2 Calmness and local error bounds

Recall that a mapping $\mathcal{M} : Y \rightrightarrows X$ between metric spaces (with both distances denoted by d) is said to be *calm* at $(\bar{y}, \bar{x}) \in \text{gph}\mathcal{M}$ if there exist a constant $\kappa \geq 0$ and neighborhoods W of \bar{x} and V of \bar{y} such that

$$d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y}) \quad (10)$$

whenever $x \in \mathcal{M}(y) \cap W$ and $y \in V$, where, as usual, the point-to-set distance $d(x, \Omega)$ is defined as $\inf \{d(x, z) : z \in \Omega\}$, and $d(x, \emptyset) := +\infty$.

The calmness property is known to be equivalent to the *metric subregularity* of the inverse multifunction $\mathcal{M}^{-1} : X \rightrightarrows Y$, given by $\mathcal{M}^{-1}(x) := \{y \in Y \mid x \in \mathcal{M}(y)\}$; the metric subregularity of \mathcal{M}^{-1} at $(\bar{x}, \bar{y}) \in \text{gph}\mathcal{M}^{-1}$ is stated in terms of the existence of a (possibly smaller) neighborhood W of \bar{x} , as well as a constant $\kappa \geq 0$, such that

$$d(x, \mathcal{M}(\bar{y})) \leq \kappa d(\bar{y}, \mathcal{M}^{-1}(x)), \text{ for all } x \in W. \quad (11)$$

An important aspect of this concept leans on the fact that the distance in the right-hand side of (11) is typically easier to compute or estimate than the distance in the left-hand side.

The infimum of all possible constants κ in (10) (over all possible combinations of κ , W , and V) is known to be equal (see, e.g., [11, Section 3H]) to the infimum of constants κ in (11) and is called the *calmness modulus* of \mathcal{M} at (\bar{y}, \bar{x}) , denoted as $\text{clm}\mathcal{M}(\bar{y}, \bar{x})$, and defined as $+\infty$ if \mathcal{M} is not calm at (\bar{y}, \bar{x}) . When dealing with the feasible set mapping (2), the calmness modulus may be seen as an enlargement rate of the feasible set around the nominal point.

Recall that an extended real-valued function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to admit a *local error bound* at $\bar{x} \in \mathbb{R}^n$ if

$$d(x, [f \leq 0]) \leq \kappa [f(x)]_+ \quad (12)$$

for a certain $\kappa \geq 0$ and for all x in a certain neighborhood W of \bar{x} , where $[f \leq 0]$ stands for $\{x \in \mathbb{R}^n : f(x) \leq 0\}$ and $[\alpha]_+$ represents the positive part of $\alpha \in \mathbb{R}$. The following result can be traced out from [4, Proposition 2.1 and Theorem 5.1] and [27, Theorem 1].

Theorem 1 *Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous proper convex function and $\bar{x} \in \mathbb{R}^n$ a point such that $f(\bar{x}) = 0$. The following conditions are equivalent:*

- (i) *f admits a local error bound at \bar{x} ;*

$$(ii) \quad \liminf_{x \rightarrow \bar{x}, f(x) > 0} d_*(0_n, \partial f(x)) > 0.$$

Moreover, under these conditions, the infimum of those $\kappa \geq 0$ satisfying (12) (for some related neighborhood W) is equal to

$$\left(\liminf_{x \rightarrow \bar{x}, f(x) > 0} d_*(0_n, \partial f(x)) \right)^{-1}.$$

3 Calmness for inequality systems with abstract constraints

The present section is devoted to compute the calmness modulus of the *feasible set mapping* \mathcal{F} in (2), where the parameter space $(\mathbb{R}^n \times \mathbb{R})^I$ is endowed with the supremum norm

$$\|(a, b)\|_\infty := \max_{t \in I} \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|,$$

with the norm in \mathbb{R}^{n+1} being defined as

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| := \max \{ \|u\|_*, |v| \} \text{ for } u \in \mathbb{R}^n, v \in \mathbb{R}. \quad (13)$$

Many results in the literature apply to the calmness of \mathcal{F} when confined to the case $C = \mathbb{R}^n$ (equivalently, $U = \emptyset$), which is the case of [9].

3.1 The case of RHS perturbations

For the sake of clarity, we start by considering the case of right-hand side (RHS, in brief) perturbations. Formally, we consider $\mathcal{F}_{\bar{a}} : \mathbb{R}^I \rightrightarrows \mathbb{R}^n$ the feasible set mapping defined by

$$\mathcal{F}_{\bar{a}}(b) := \mathcal{F}(\bar{a}, b) \text{ for all } b \in \mathbb{R}^I;$$

where $\bar{a} \in (\mathbb{R}^n)^I$ remains fixed and $b \in \mathbb{R}^I$ is the parameter to be perturbed.

In the case when $C = \mathbb{R}^n$ it is well-known that $\mathcal{F}_{\bar{a}}$ is always calm at any point of its (polyhedral) graph as a consequence of a classical result by Robinson [36]. Observe that this is not the case for the more general abstract/set constrained setting of systems (1) with C being a closed convex set, not polyhedral. Just consider, for instance, the example when C is the closed unit ball in \mathbb{R}^2 and $\bar{a} = (1, 0)'$, $\mathcal{F}_{\bar{a}}(b) := \{x \in C; x_1 \leq b\}$. If we consider the nominal elements $\bar{b} = -1$, $\bar{x} = (-1, 0)'$, one easily sees that $\mathcal{F}_{\bar{a}}(b)$ is not calm at (\bar{b}, \bar{x}) .

Moreover, in the particular case in which $C = \mathbb{R}^n$, the calmness of $\mathcal{F}_{\bar{a}}$ at $(\bar{b}, \bar{x}) \in \text{gph} \mathcal{F}_{\bar{a}}$ can be easily translated into the local error bound property of the max-function defined by $x \mapsto \max_{t \in I} [\bar{a}'_t x - \bar{b}_t]_+$.

Coming back to the general case when C is a closed convex set, associated with $(\bar{b}, \bar{x}) \in \text{gph} \mathcal{F}_{\bar{a}}$ we consider the proper convex lsc function $s_{\bar{b}} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$s_{\bar{b}}(x) := \max_{t \in I} (\bar{a}'_t x - \bar{b}_t) + \mathbf{I}_C(x). \quad (14)$$

Obviously

$$[s_{\bar{b}} \leq 0] = \mathcal{F}_{\bar{a}}(\bar{b}) = \mathcal{F}(\bar{a}, \bar{b}).$$

Observe that the specification of (11) defining the metric subregularity of $\mathcal{F}_{\bar{a}}^{-1}$ at (\bar{x}, \bar{b}) turns out to be equivalent to

$$d(x, [s_{\bar{b}} \leq 0]) \leq \kappa [s_{\bar{b}}(x)]_+ \quad (15)$$

for a certain $\kappa \geq 0$ and for all x in a certain neighborhood W of \bar{x} . Note that (15), which holds trivially if $x \notin C$, is nothing else but the existence of a local error bound for $s_{\bar{b}}$ at \bar{x} ; see (12).

Remark 1 If $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$ satisfies $s_{\bar{b}}(\bar{x}) < 0$, then $\text{clm} \mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = 0$, because the continuity of the max-function appearing in (14) ensures the existence of a neighborhood W of \bar{x} such that

$$s_{\bar{b}}(x) < 0, \text{ for all } x \in W \cap C.$$

So (15) trivially holds for all $\kappa > 0$ and all $x \in W$, either $x \in W \cap C$ or $x \in W \setminus C$ (where both sides of (15) are $+\infty$).

So, from now on we deal with the non-trivial case $s_{\bar{b}}(\bar{x}) = 0$. Therefore, we may apply Theorem 1 to conclude the following result which constitutes the starting point in our analysis:

Proposition 1 For $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$, with $s_{\bar{b}}(\bar{x}) = 0$, we have

$$\text{clm} \mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = \left(\liminf_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} d_*(0, \partial s_{\bar{b}}(x)) \right)^{-1}.$$

The rest of this section is devoted to translate the previous proposition into a point-based formula for the calmness modulus of $\mathcal{F}_{\bar{a}}$, as an extension

of [9, Theorem 4] to the case when $C \neq \mathbb{R}^n$. To start with, one easily checks the following equality

$$\liminf_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} d_*(0_n, \partial s_{\bar{b}}(x)) = d_* \left(0_n, \limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(x) \right).$$

In this way, this subsection mainly consists of providing a point-based formula for the outer limit of subdifferentials $\limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(x)$, where this limit of sets is understood in the Painlevé-Kuratowski sense. The reader is addressed to [8, Theorem 3.1] for the particular case when $C = \mathbb{R}^n$.

If we denote

$$m_{\bar{b}}(x) := \max_{t \in I} (\bar{a}'_t x - \bar{b}_t),$$

then clearly $\text{dom } m_{\bar{b}} = \mathbb{R}^n$ and since $\text{ri}C \neq \emptyset$, we can apply (5), (6), and (8), together with the well-known Valadier formula, to get

$$\partial s_{\bar{b}}(x) = \emptyset \text{ if } x \notin C,$$

and

$$\begin{aligned} \partial s_{\bar{b}}(x) &= \partial m_{\bar{b}}(x) + \partial \mathbf{I}_C(x) \\ &= \text{conv} \{ \bar{a}_t \mid t \in I_{\bar{b}}(x) \} + \mathbf{N}_C(x), \text{ for all } x \in C, \end{aligned}$$

where

$$I_{\bar{b}}(x) := \{ t \in I \mid \bar{a}'_t x - \bar{b}_t = m_{\bar{b}}(x) \},$$

which is trivially nonempty. Moreover, (7) yields

$$s'_{\bar{b}}(x; d) = m'_{\bar{b}}(x; d) + \mathbf{I}_{T_C(x)}(d), \text{ for all } d \in \mathbb{R}^n, \quad (16)$$

and from (16) one immediately concludes

$$\begin{aligned} D(x) &:= \left\{ d \in \mathbb{R}^n \mid 0 < s'_{\bar{b}}(x; d) < +\infty \right\} \\ &= \{ d \in T_C(x) \mid m'_{\bar{b}}(x; d) > 0 \}. \end{aligned}$$

Now we associate with each $d \in D(x)$ the sets

$$I(x; d) := \{ t \in I_{\bar{b}}(x) \mid \bar{a}'_t d = m'_{\bar{b}}(x; d) \} \quad \text{and} \quad \mathbf{N}_C(x) \cap \{d\}^\perp.$$

Obviously $I(x; d)$ is non-empty, while $\mathbf{N}_C(x) \cap \{d\}^\perp$ may collapse to the origin. It is also immediate, from (16) and the definition of $I(x; d)$ that, for all $d \in D(x)$, the hyperplane

$$H_{x,d} := \{ z \in \mathbb{R}^n \mid d'z = m'_{\bar{b}}(x; d) \},$$

supports $\partial s_{\bar{b}}(x)$ and

$$H_{x,d} \cap \partial s_{\bar{b}}(x) = \text{conv} \{ \bar{a}_t, t \in I(x; d) \} + \left(N_C(x) \cap \{d\}^\perp \right). \quad (17)$$

From now on in this section, our results are established under the additional assumption that C is *locally polyhedral*, i.e.

$$\mathbb{R}_+(C - x) \text{ is closed for all } x \in C.$$

So, in this case any $0_n \neq d \in T_C(x)$ is a feasible direction of C at x since $T_C(x) = \text{cl} \mathbb{R}_+(C - x)$.

Lemma 3.2 in [2] proves that if C is *locally polyhedral*, then $\mathbb{R}_+(C - x)$ is polyhedral at every $x \in C$, and so, $N_C(x)$ is also polyhedral at every $x \in C$.

Lemma 2 *Consider $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$ with $s_{\bar{b}}(\bar{x}) = 0$, and assume that C is locally polyhedral. For any $d \in D(\bar{x})$ there exists $\bar{\lambda}_d > 0$ such that*

$$\bar{x} + \lambda d \in C, \quad m_{\bar{b}}(\bar{x} + \lambda d) = \lambda m'_{\bar{b}}(\bar{x}; d) \quad \text{and} \quad I_{\bar{b}}(\bar{x} + \lambda d) = I(\bar{x}; d),$$

whenever $0 < \lambda < \bar{\lambda}_d$.

Proof. Take $\tilde{\lambda}_d > 0$ such that

$$\bar{x} + \lambda d \in C \text{ and } I_{\bar{b}}(\bar{x} + \lambda d) \subset I_{\bar{b}}(\bar{x}), \text{ whenever } 0 < \lambda < \tilde{\lambda}_d \quad (18)$$

(remember that $D(\bar{x}) \subset T_C(\bar{x})$, while the inclusion $I_{\bar{b}}(\bar{x} + \lambda d) \subset I_{\bar{b}}(\bar{x})$ for sufficiently small λ comes from the continuity of $m_{\bar{b}}$ at \bar{x}).

Now, let us show the existence of $0 < \bar{\lambda}_d \leq \tilde{\lambda}_d$ such that

$$m_{\bar{b}}(\bar{x} + \lambda d) = \lambda m'_{\bar{b}}(\bar{x}; d), \text{ whenever } 0 < \lambda < \bar{\lambda}_d. \quad (19)$$

Taking $0 < \bar{\lambda}_d \leq \tilde{\lambda}_d$ such that

$$\bar{\lambda}_d \max_{t \in I \setminus I_{\bar{b}}(\bar{x})} \{ \|\bar{a}_t\|_* \|d\| \} \leq \inf_{t \in I \setminus I_{\bar{b}}(\bar{x})} \{ \bar{b}_t - \bar{a}'_t \bar{x} \},$$

then, one can easily check that

$$\max_{t \in I \setminus I_{\bar{b}}(\bar{x})} \{ (\bar{a}'_t \bar{x} - \bar{b}_t) + \lambda \bar{a}'_t d \} \leq 0, \text{ for any } 0 < \lambda < \bar{\lambda}_d.$$

Observing that $s_{\bar{b}}(\bar{x}) = m_{\bar{b}}(\bar{x}) = 0$, we get for $0 < \lambda < \bar{\lambda}_d$,

$$\begin{aligned} m_{\bar{b}}(\bar{x} + \lambda d) &= \max_{t \in I} \{ (\bar{a}'_t \bar{x} - \bar{b}_t) + \lambda \bar{a}'_t d \} \\ &= \max_{t \in I_{\bar{b}}(\bar{x})} \{ (\bar{a}'_t \bar{x} - \bar{b}_t) + \lambda \bar{a}'_t d \} = \lambda m'_{\bar{b}}(\bar{x}; d). \end{aligned}$$

Finally, let us prove

$$I_{\bar{b}}(\bar{x} + \lambda d) = I(\bar{x}; d), \text{ for } 0 < \lambda < \bar{\lambda}_d.$$

Take any $0 < \lambda < \bar{\lambda}_d$. If $t_0 \in I_{\bar{b}}(\bar{x} + \lambda d)$, then from (18) and (19) we have

$$\bar{a}'_{t_0} d = \frac{1}{\lambda} (\bar{a}'_{t_0} (\bar{x} + \lambda d) - \bar{b}_{t_0}) = \frac{1}{\lambda} m_{\bar{b}}'(\bar{x} + \lambda d) = m_{\bar{b}}'(\bar{x}; d),$$

which yields $t_0 \in I(\bar{x}; d)$. Reciprocally, if $t_0 \in I(\bar{x}; d)$, then

$$\bar{a}'_{t_0} (\bar{x} + \lambda d) - \bar{b}_{t_0} = \lambda m_{\bar{b}}'(\bar{x}; d) = m_{\bar{b}}'(\bar{x} + \lambda d).$$

■

Theorem 2 *Let $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$ with $s_{\bar{b}}(\bar{x}) = 0$, and assume that C is locally polyhedral. For any $d \in D(\bar{x})$, there exists $\bar{\lambda}_d > 0$ such that*

$$\partial s_{\bar{b}}(\bar{x} + \lambda d) = \text{conv} \{ \bar{a}_t, t \in I(\bar{x}; d) \} + \left(N_C(\bar{x}) \cap \{d\}^\perp \right), \text{ for all } 0 < \lambda < \bar{\lambda}_d.$$

Proof. Fix any $d \in D(\bar{x})$ and take $\bar{\lambda}_d > 0$ verifying the statement of the previous lemma; then, for any $0 < \lambda < \bar{\lambda}_d$ we have

$$\bar{x} + \lambda d \in C, \quad I_{\bar{b}}(\bar{x} + \lambda d) = I(\bar{x}; d), \quad \text{and} \quad N_C(\bar{x} + \lambda d) = N_C(\bar{x}) \cap \{d\}^\perp,$$

where the last equality comes from Lemma 1. Then, for $0 < \lambda < \bar{\lambda}_d$ we have

$$\begin{aligned} \partial s_{\bar{b}}(\bar{x} + \lambda d) &= \text{conv} \{ \bar{a}_t, t \in I_{\bar{b}}(\bar{x} + \lambda d) \} + N_C(\bar{x} + \lambda d) \\ &= \text{conv} \{ \bar{a}_t, t \in I(\bar{x}; d) \} + \left(N_C(\bar{x}) \cap \{d\}^\perp \right). \end{aligned}$$

The following lemma is a direct consequence of [2, Lemma 3.1]. ■

Lemma 3 *Assume that C is locally polyhedral and let $\{x^r\} \subset C$ converging to \bar{x} with $x^r \neq \bar{x}$ for all r . Then, there exists $r_0 \in \mathbb{N}$ and a sequence of scalars $\{\lambda_r\}_{r \geq r_0} \subset]1, +\infty[$ such that $\bar{x} + \lambda_r(x^r - \bar{x}) \in C$, for all $r \geq r_0$.*

Theorem 3 *Consider $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$ with $s_{\bar{b}}(\bar{x}) = 0$, and assume that C is locally polyhedral. We have*

$$\begin{aligned} \limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(x) &= \bigcup_{d \in D(\bar{x})} \left\{ \text{conv} \{ \bar{a}_t, t \in I(\bar{x}; d) \} + \left(N_C(\bar{x}) \cap \{d\}^\perp \right) \right\} \\ &= \bigcup_{d \in D(\bar{x})} \limsup_{\lambda \downarrow 0} \partial s_{\bar{b}}(\bar{x} + \lambda d). \end{aligned}$$

Proof. The second equality comes straightforwardly from the previous theorem. Moreover, it is immediate that

$$\bigcup_{d \in D(\bar{x})} \limsup_{\lambda \downarrow 0} \partial s_{\bar{b}}(\bar{x} + \lambda d) \subset \limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(\bar{x}).$$

Just observe that for any $d \in D(\bar{x})$ Lemma 2 ensures the existence of $\bar{\lambda}_d > 0$ such that

$$s_{\bar{b}}(\bar{x} + \lambda d) = m_{\bar{b}}(\bar{x} + \lambda d) = \lambda m'_{\bar{b}}(\bar{x}; d) > 0, \text{ for } 0 < \lambda < \bar{\lambda}_d.$$

So, it remains to prove the inclusion

$$\limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(x) \subset \bigcup_{d \in D(\bar{x})} \left\{ \text{conv} \{ \bar{a}_t, t \in I(\bar{x}; d) \} + \left(N_C(\bar{x}) \cap \{d\}^\perp \right) \right\}.$$

Let

$$u \in \limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(x),$$

and write $u = \lim_r u^r$, with

$$u^r \in \partial s_{\bar{b}}(x^r) = \text{conv} \{ \bar{a}_t; t \in I_{\bar{b}}(x^r) \} + N_C(x^r), \text{ for all } r,$$

for some sequence $\{x^r\}$ converging to \bar{x} such that $s_{\bar{b}}(x^r) > 0$ for all r . In particular, $\partial s_{\bar{b}}(x^r) \neq \emptyset$ for all r entails $\{x^r\} \subset C$. Moreover, $s_{\bar{b}}(x^r) > 0$ implies $x^r \neq \bar{x}$ for all r .

The finiteness of I allows us to assume (by taking a subsequence if needed) that $\{I_{\bar{b}}(x^r)\}$ is constant, say $I_{\bar{b}}(x^r) = D$ for all r . In particular, $D \subset I_{\bar{b}}(\bar{x})$.

Moreover, the previous lemma ensures the existence of $\lambda_r \in]1, +\infty[$ such that $\bar{x} + \lambda_r(x^r - \bar{x}) \in C$ for r large enough (say for all r , without loss of generality). In this way, from Lemma 1 we conclude

$$N_C(x^r) = N_C(\bar{x}) \cap \{x^r - \bar{x}\}^\perp, \text{ for all } r.$$

Since $N_C(\bar{x})$ is polyhedral and $N_C(\bar{x}) \cap \{x^r - \bar{x}\}^\perp$ is a face of $N_C(\bar{x})$, there is a finite amount of possibilities. Consequently, we may assume that $N_C(\bar{x}) \cap \{x^r - \bar{x}\}^\perp = N_C(\bar{x}) \cap \{x^{r_0} - \bar{x}\}^\perp$ for some fixed $r_0 \in \mathbb{N}$.

Then, define

$$d^0 := \frac{x^{r_0} - \bar{x}}{s_{\bar{b}}(x^{r_0})}.$$

Let us see that $d^0 \in D(\bar{x})$. Clearly, $d^0 \in T_C(\bar{x})$ and

$$\begin{aligned} m'_b(\bar{x}; d^0) &= \max_{t \in I_{\bar{b}}(\bar{x})} \frac{\bar{a}'_t(x^{r_0} - \bar{x})}{s_{\bar{b}}(x^{r_0})} \\ &= \max_{t \in I_{\bar{b}}(\bar{x})} \frac{\bar{a}'_t x^{r_0} - \bar{b}_t}{s_{\bar{b}}(x^{r_0})} \\ &= \max_{t \in D} \frac{\bar{a}'_t x^{r_0} - \bar{b}_t}{s_{\bar{b}}(x^{r_0})} = 1. \end{aligned}$$

In fact,

$$\frac{\bar{a}'_t x^{r_0} - \bar{b}_t}{s_{\bar{b}}(x^{r_0})} < 1 \text{ if } t \in I_{\bar{b}}(\bar{x}) \setminus D,$$

and

$$\frac{\bar{a}'_t x^{r_0} - \bar{b}_t}{s_{\bar{b}}(x^{r_0})} = 1 \text{ if } t \in D.$$

So, $d^0 \in D(\bar{x})$ and $I(\bar{x}; d^0) = D$.

In summary

$$\begin{aligned} u^r &\in \text{conv}\{\bar{a}_t; t \in I_{\bar{b}}(x^r)\} + N_C(x^r) \\ &= \text{conv}\{\bar{a}_t; t \in D\} + \left(N_C(\bar{x}) \cap \{x^{r_0} - \bar{x}\}^\perp\right) \\ &= \text{conv}\{\bar{a}_t; t \in D\} + \left(N_C(\bar{x}) \cap \{d^0\}^\perp\right), \end{aligned}$$

with $d^0 \in D(\bar{x})$, which entails

$$u \in \text{conv}\{\bar{a}_t; t \in D\} + \left(N_C(\bar{x}) \cap \{d^0\}^\perp\right).$$

■

Corollary 1 *Consider $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$ with $s_{\bar{b}}(\bar{x}) = 0$, and assume that C is locally polyhedral. We have*

$$\text{clm}\mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = \left(\min_{d \in D(\bar{x})} d_* \left(0_n, \text{conv}\{\bar{a}_t, t \in I(\bar{x}; d)\} + \left(N_C(\bar{x}) \cap \{d\}^\perp\right) \right) \right)^{-1}.$$

Remark 2 (i) Observe that when C is locally polyhedral, $\mathcal{F}_{\bar{a}}$ is calm at all $(\bar{b}, \bar{x}) \in \text{gph}\mathcal{F}_{\bar{a}}$. In fact, from (17) we know that $0_n \notin \text{conv}\{\bar{a}_t, t \in I(\bar{x}; d)\} + (N_C(\bar{x}) \cap \{d\}^\perp)$. Moreover, the ‘locally polyhedral’ assumption entails the finiteness of possibilities when we are taking the minimum in the previous corollary. Finally, as commented above, $\text{clm}\mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = 0$ when $s_{\bar{b}}(\bar{x}) < 0$.

(ii) The second equality in the previous theorem brings to mind the notion of directional limiting subdifferential. In fact, for each direction $d \in \mathbb{R}^n$, the set $\limsup_{\lambda \downarrow 0} \partial s_{\bar{b}}(\bar{x} + \lambda d)$ is clearly included in the analytic limiting subdifferential of $s_{\bar{b}}$ in the direction of d , denoted by $\partial_d s_{\bar{b}}(\bar{x})$; see [5, p. 5] for the definition in \mathbb{R}^n and [14] for a previous reference in Banach spaces (the reader is also addressed to [13] for related topics as the directional metric regularity and subregularity). In fact, one could possibly employ this notion to avoid the ‘locally polyhedral’ assumption. Specifically, the role played by $\bigcup_{d \in D(\bar{x})} \partial_d s_{\bar{b}}(\bar{x})$ when C is an arbitrary closed convex set could constitute a matter of further research. In any case, the polyhedrality assumption is not restrictive for our purposes of applying the calmness results to the analysis of the convergence of the central path, and it has some advantages from the practical point of view (due to the finiteness of possibilities in the previous minimum, as commented above).

3.2 The polyhedral case

If C is a polyhedral set, expressed explicitly by means of the linear inequalities

$$C = \{x \in \mathbb{R}^n \mid \bar{c}'_s x \leq \bar{d}_s, \ s \in U\}, \quad (20)$$

with U being a finite non-empty index set, disjoint of I , and $(\bar{c}_s, \bar{d}_s)_{s \in U}$ fixed (i.e., inequalities indexed by elements of U are not subject to perturbations), with $\bar{c}_s \neq 0_n$ for all $s \in U$, we have

$$\mathcal{F}_{\bar{a}}(b) := \{x \in \mathbb{R}^n \mid \bar{c}'_s x \leq \bar{d}_s, \ s \in U; \ \bar{a}'_t x \leq b_t, \ t \in I\}. \quad (21)$$

In this particular case for any $x \in C$ we have (recall (9)):

$$N_C(x) = \text{cone} \{\bar{c}_s; \ s \in U(x)\},$$

where

$$U(x) := \{s \in U : \bar{c}'_s x - \bar{d}_s = 0\}.$$

The following theorem constitutes a specification of Theorem 3 to our current polyhedral setting. Observe that this new expression for the outer limit of subdifferentials has the virtue of being conceptually implementable as far as it only involves the nominal elements and a finite family of pairs of subsets of indices which is defined as follows:

Associated with $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$, let us define the family $\mathcal{I}(\bar{x})$ formed by all pairs of subsets of indices of the form (I_1, U_1) with $I_1 \subset I_{\bar{b}}(\bar{x})$, $U_1 \subset U(\bar{x})$, and such that the following system has a solution in d :

$$\begin{cases} m'_b(\bar{x}; d) > 0, \\ \bar{a}'_t d = m'_b(\bar{x}; d), \quad t \in I_1, \\ \bar{a}'_t d < m'_b(\bar{x}; d), \quad t \in I_{\bar{b}}(\bar{x}) \setminus I_1, \\ \bar{c}'_s d = 0, \quad s \in U_1, \\ \bar{c}'_s d < 0, \quad s \in U(\bar{x}) \setminus U_1. \end{cases} \quad (22)$$

Theorem 4 Consider $\bar{x} \in \mathcal{F}_{\bar{a}}(\bar{b})$ with $s_{\bar{b}}(\bar{x}) = 0$, and assume that C is the polyhedral set (20). Then,

$$\limsup_{x \rightarrow \bar{x}, s_{\bar{b}}(x) > 0} \partial s_{\bar{b}}(x) = \bigcup_{(I_1, U_1) \in \mathcal{I}(\bar{x})} (\text{conv} \{\bar{a}_t, t \in I_1\} + \text{cone} \{\bar{c}_s, s \in U_1\}).$$

Consequently

$$\text{clm} \mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = \left(\min_{(I_1, U_1) \in \mathcal{I}(\bar{x})} d_*(0_n, \text{conv} \{\bar{a}_t, t \in I_1\} + \text{cone} \{\bar{c}_s, s \in U_1\}) \right)^{-1}.$$

Proof. The inclusion ‘ \subset ’ comes from applying Theorem 3, just by observing that if $d \in D(\bar{x})$, then $m'_b(\bar{x}; d) > 0$, and $d \in T_C(\bar{x})$, which implies $\bar{c}'_s d \leq 0$ for all $s \in U(\bar{x})$. Moreover, $I_1 := I(\bar{x}; d) \subset I_{\bar{b}}(\bar{x})$ together with

$$U_1 := \{s \in U(\bar{x}) \mid \bar{c}'_s d = 0\},$$

verifies that $(I_1, U_1) \in \mathcal{I}(\bar{x})$. Finally, observe that

$$N_C(\bar{x}) \cap \{d\}^\perp = \text{cone} \{\bar{c}_s, s \in U_1\}.$$

For the reciprocal inclusion, consider any $(I_1, U_1) \in \mathcal{I}(\bar{x})$ and take $d \in \mathbb{R}^n$ as a solution of system (22). Then, $d \in D(\bar{x})$, $I(\bar{x}; d) = I_1$ and again $\text{cone} \{\bar{c}_s, s \in U_1\} = N_C(\bar{x}) \cap \{d\}^\perp$. ■

Remark 3 In the previous theorem we could confine ourselves to those $(I_1, U_1) \in \mathcal{I}(\bar{x})$ which are maximal with respect to the coordinatewise inclusion order.

The following examples illustrate the difference between the contexts of total and partial perturbations of the RHS.

Example 1 Let us consider the system, in \mathbb{R}^2 endowed with the Euclidean norm, given by

$$\left\{ \begin{array}{ll} x_1 \leq b, & t = 1 \in I \\ -x_1 - x_2 \leq 0, & s = 2 \in U \end{array} \right\};$$

For $\bar{b} = 0$ and $\bar{x} = (0, 0)'$, we have $\mathcal{I}(\bar{x}) = \{(\{1\}, \emptyset), (\{1\}, \{2\})\}$ and

$$\text{clm}\mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = (d_*(0_2, \text{conv}\{(1, 0)'\} + \text{cone}\{(-1, -1)'\}))^{-1} = \sqrt{2}.$$

Observe that in the framework of perturbations of the whole RHS, i.e., the case $I = \{1, 2\}$, $U = \emptyset$, and $\bar{b} = (0, 0)'$, the corresponding calmness modulus, according to [9, Theorem 4], is equal to $(d_*(0_2, \text{conv}\{(1, 0)', (-1, -1)'\}))^{-1} = \sqrt{5}$.

Example 2 Let us consider the system, in \mathbb{R}^2 endowed with the Euclidean norm, given by

$$\left\{ \begin{array}{ll} -x_1 - x_2 \leq b, & t = 1 \in I \\ x_1 \leq 0, & s = 2 \in U \\ x_2 \leq 0, & s = 3 \in U \end{array} \right\};$$

For $\bar{b} = 0$ and $\bar{x} = (0, 0)'$, we have

$$\mathcal{I}(\bar{x}) = \{(\{1\}, \emptyset), (\{1\}, \{2\}), (\{1\}, \{3\})\} \text{ and } \text{clm}\mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) = 1,$$

whereas in the framework of perturbations of the whole RHS, again according to [9, Theorem 4], the calmness modulus equals $\sqrt{5}$.

3.3 Perturbations of all coefficients

The development of this subsection is very similar to that of [9, Section 5]. To start with, for any $x \in \mathbb{R}^n$, recall that $d(\bar{b}, \mathcal{F}_{\bar{a}}^{-1}(x)) = [s_{\bar{b}}(x)]_+$, while, following the argument of [7, Lem. 10], one has

$$d((\bar{a}, \bar{b}), \mathcal{F}^{-1}(x)) = \left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\|_*^{-1} [s_{\bar{b}}(x)]_+.$$

For completeness purposes, let us recall that the previous expression comes from applying the well-known Ascoli formula for the distance from $(\frac{\bar{a}_t}{\bar{b}_t})$ to the half-space $\{w \in \mathbb{R}^{n+1} \mid (\frac{x}{-1})'w \leq 0\}$. Moreover, with our current choice of norms (13),

$$\left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\|_* = \|x\| + 1,$$

and, so,

$$d((\bar{a}, \bar{b}), \mathcal{F}^{-1}(x)) = (\|\bar{x}\| + 1) d(\bar{b}, \mathcal{F}_{\bar{a}}^{-1}(x)).$$

Then, in a completely analogous way to [9, Theorem 5] (see also [9, Remark 10]), one can prove the first equality of the following result, whereas the second equality is a straightforward consequence of Theorem 3.

Theorem 5 *Assume that C is locally polyhedral. For $\bar{x} \in \mathcal{F}(\bar{a}, \bar{b})$ we have*

$$\begin{aligned} \text{clm}\mathcal{F}((\bar{a}, \bar{b}), \bar{x}) &= (\|\bar{x}\| + 1) \text{clm}\mathcal{F}_{\bar{a}}(\bar{b}, \bar{x}) \\ &= \frac{\|\bar{x}\| + 1}{\min_{d \in D(\bar{x})} d_*(0_n, \text{conv}\{\bar{a}_t, t \in I(\bar{x}; d)\} + (N_C(\bar{x}) \cap \{d\}^\perp))}. \end{aligned}$$

4 On the convergence behavior of the central path

We consider the family of nonlinear problems $\{(P_\mu)\}_{\mu > 0}$ defined in (3). A standard reformulation of the KKT conditions (see, e.g., [32]) applied to (P_μ) gives rise to the following non-linear system in the variable $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$

$$\begin{aligned} c - A'y - z &= 0_n, \\ Ax &= b, \\ x_i z_i &= \mu, i = 1, \dots, n, \\ x, z &\geq 0_n. \end{aligned} \tag{23}$$

Observe that any solution (x, y, z) of (23) satisfies $x, z > 0_n$, since $x_i z_i = \mu > 0$ for all i .

On the other hand, the well-known KKT conditions for the original LP problem (3) read as,

$$\begin{aligned} c - A'y - z &= 0_n, \\ Ax &= b, \\ x_i z_i &= 0, i = 1, \dots, n \\ x, z &\geq 0_n. \end{aligned} \tag{24}$$

From now on, let $\Lambda \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ denote the set of all solutions of (24). It is well-known that

$$\Lambda = S(P) \times S(D),$$

where $S(D)$ denotes the optimal set of (D) , the dual problem of (P) .

We work under the following assumptions (which are equivalent to the fulfilment of Assumption 2.1(a, b, and c) in [32]):

- The set of optimal solutions of (P) , which we denote by $S(P)$, is non-empty and bounded.

- The Slater constraint qualification (SCQ) holds at (P) ; i.e., there exists a Slater point $\hat{x} > 0_n$ (i.e. all coordinates positive) satisfying $A\hat{x} = b$ and the rank of A is m ($\leq n$).

The following proposition can be traced out from Propositions 2.1, 2.2 and 2.3 in [32].

Proposition 2 *Under the current assumptions, we have:*

(i) *For each $\mu > 0$, problem (P_μ) has a unique optimal solution, say $x(\mu)$; moreover, there exists $(y(\mu), z(\mu)) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $(x(\mu), y(\mu), z(\mu))$ is the unique solution of the KKT system (23).*

(ii) *Keeping the previous notation, we have that*

$$\lim_{\mu \rightarrow 0} (x(\mu), y(\mu), z(\mu)) = (x^0, y^0, z^0) \in \Lambda,$$

where x^0 is an optimal solution of (P) and (y^0, z^0) is an optimal solution of its dual, i.e. $x^0 \in S(P)$ and $(y^0, z^0) \in S(D)$.

In this framework, a typical computation yields

$$c'x(\mu) - b'y(\mu) = x(\mu)'z(\mu) = n\mu,$$

and this quantity is the so-called *duality gap* at $(x(\mu), y(\mu), z(\mu))$.

For each $\mu > 0$, $(x(\mu), y(\mu), z(\mu))$ is usually obtained by applying the classical Newton method to the KKT system mentioned above. This is the so-called *primal-dual path following method*, which is widely considered to be a notably efficient interior point method. Recall that the interest in interior point methods comes from the work of Karmarkar [23], where the first interior point algorithm with polynomial time complexity was introduced. One can find in the literature particular implementations of this generic scheme, such as the pioneering works [35] and [32]. See also [1] for a different implementation, coming from a specific reduced KKT system. The reader is addressed to [16] and [34] for comprehensive surveys on the field of interior point methods. Borrowing the notation of Monteiro and Adler [32], standard convergence results for interior point methods are focused on the behavior of $\|f(x, z) - \mu e\|_2$, where $f(x, z) \in \mathbb{R}^n$ has components $x_i z_i$, $i = 1, \dots, n$, given that x and z yield the duality gap $n\mu = x'z$. Being a scalar measure associated with two vectors, driving the duality gap to zero is not sufficient to ensure convergence: some componentwise measure of proximity to the central path must also be reduced sufficiently in each

iteration. For Monteiro and Adler [32], this is characterized by the condition that $\|f(x, z) - \mu e\|_2 \leq \theta \mu$, for some constant $\theta \in [0, \frac{1}{2})$.

Here, we introduce a certain feasible set mapping which will allow us to analyze the speed of convergence of $(x(\mu), y(\mu), z(\mu))$ to Λ via the analysis of the calmness modulus of this mapping. Let $(x^0, y^0, z^0) \in \Lambda$ be as in the previous proposition, consider

$$I^0 := \{i = 1, \dots, n \mid x_i^0 = 0\}$$

and define the multifunction $\mathcal{F}^0 : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ given by

$$\mathcal{F}^0(u) := \left\{ (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \left| \begin{array}{l} c - A'y - z = 0_n, \\ Ax = b, \ x \geq 0_n, \ z \geq 0_n, \\ x_i \leq u_i, \ i \in I^0 \\ z_j \leq u_j, \ j \in \{1, \dots, n\} \setminus I^0 \end{array} \right. \right\}, \quad (25)$$

which is nothing else but a feasible set mapping associated with a linear system of inequalities and equations parametrized with respect to the right-hand side of a specific block of inequality constraints. In this setting, we are considering the space of parameters, \mathbb{R}^n , endowed with the supremum norm, $\|\cdot\|_\infty$, and the space of variables, $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$, with any norm satisfying

$$\|(x, y, z)\| \geq \max\{\|x\|_\infty, \|z\|_\infty\}. \quad (26)$$

For instance, any p -norm satisfies this property.

The following theorem provides a certain measure of the convergence of the central path with respect to parameter μ . Considering (x^0, y^0, z^0) as in Proposition 2, define

$$\alpha_0 := \min\{x_i^0 + z_i^0, \ i = 1, \dots, n\} \text{ and } \beta_0 := \max\{x_i^0 + z_i^0, \ i = 1, \dots, n\}. \quad (27)$$

These scalars α_0 and β_0 are inspired by the concept of *condition number* of a problem, which follows from [39, Definition 16].

Remark 4 *It is well-known in the context of LP problems, that under the current assumptions ($\emptyset \neq S(P)$ bounded and SCQ), that (x^0, y^0, z^0) is non-degenerate;*

$$\text{i.e., } 0 < x_i^0 + z_i^0, \text{ for all } i = 1, \dots, n,$$

which obviously entails

$$0 < \alpha_0 (\leq \beta_0).$$

Theorem 6 Assume that $\emptyset \neq S(P)$ is bounded and the SCQ holds. Consider the central path of (P) , $\{(x(\mu), y(\mu), z(\mu)); \mu > 0\}$, and its limits point (x^0, y^0, z^0) . Then:

(i) We have that, with the notation (27),

$$\text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0)) \geq \frac{\alpha_0}{\beta_0} > 0. \quad (28)$$

(ii) For any

$$\kappa > \alpha_0^{-1} \text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0))$$

there exists $\varepsilon > 0$ such that

$$d((x(\mu), y(\mu), z(\mu)), \Lambda) \leq \kappa\mu, \text{ whenever } 0 < \mu < \varepsilon.$$

Proof. (i) Let us see that for any α, β , and γ such that $0 < \alpha < \alpha_0$, $\beta > \beta_0$, and

$$\gamma > \text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0)),$$

one has

$$\gamma \geq \frac{\alpha}{\beta},$$

which allows us to conclude that our desired inequality (28) holds.

Take α, β , and γ as above and consider any sequence of positive scalars $\{\mu_r\}_r$ converging to 0 and define the following sequences of parameters:

$$u^r := \frac{\mu_r}{\alpha} e \in \mathbb{R}^n, \quad r \in \mathbb{N},$$

where $e := (1, 1, \dots, 1)' \in \mathbb{R}^n$. Associated with $\{\mu_r\}_r$, let us consider the corresponding sequence of elements in the central path $\{(x(\mu_r), y(\mu_r), z(\mu_r))\}$, which converges to (x^0, y^0, z^0) by assumption. So, for each r , we have

$$x_i(\mu_r) z_i(\mu_r) = \mu_r, \quad i = 1, \dots, n.$$

Note that, by assumption,

$$\begin{cases} z_i^0 = x_i^0 + z_i^0 \geq \alpha_0 (> \alpha), & \text{for } i \in I^0, \\ x_i^0 = x_i^0 + z_i^0 \geq \alpha_0 (> \alpha), & \text{for } \{1, \dots, n\} \setminus I^0. \end{cases}$$

Them, for r large enough, say $r \geq r_0$,

$$\begin{cases} \frac{\mu_r}{\beta} \leq x_i(\mu_r) = \frac{\mu_r}{z_i(\mu_r)} \leq \frac{\mu_r}{\alpha}, & \text{for all } i \in I^0, \\ \frac{\mu_r}{\beta} \leq z_i(\mu_r) = \frac{\mu_r}{x_i(\mu_r)} \leq \frac{\mu_r}{\alpha}, & \text{for all } i \in \{1, \dots, n\} \setminus I^0. \end{cases} \quad (29)$$

Consequently,

$$(x(\mu_r), y(\mu_r), z(\mu_r)) \in \mathcal{F}^0(u^r), \text{ for } r \geq r_0.$$

Then, the choice of $\gamma (> \text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0)))$ guarantees the existence of $r_1 \geq r_0$ such that

$$d((x(\mu_r), y(\mu_r), z(\mu_r)), \mathcal{F}^0(0_n)) \leq \gamma d(u^r, 0_n) = \gamma \frac{\mu_r}{\alpha}, \text{ for all } r \geq r_1. \quad (30)$$

On the other hand,

$$\mathcal{F}^0(0_n) \subset \{(x, y, z) \mid x_i = 0, i \in I^0, z_i = 0, i \in \{1, \dots, n\} \setminus I^0\}$$

and, then,

$$d((x(\mu_r), y(\mu_r), z(\mu_r)), \mathcal{F}^0(0_n)) \geq \frac{\mu_r}{\beta}, \text{ } r \geq r_0, \quad (31)$$

where we have taken (26) into account and applied the lower bounds for $x_i(\mu_r)$, $i \in I^0$, and $z_i(\mu_r)$, $i \in \{1, \dots, n\} \setminus I^0$ given in (29). Finally, from (30) and (31) we conclude

$$\frac{\mu_r}{\beta} \leq \gamma \frac{\mu_r}{\alpha}, \text{ for all } r \geq r_1,$$

which yields $\gamma \geq \frac{\alpha}{\beta}$, as required.

(ii) Let $\kappa_1 > \alpha_0^{-1} \text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0))$ and consider

$$0 < \alpha_1 := \kappa_1^{-1} \text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0)) < \alpha_0$$

Consider any α_2 satisfying $\alpha_1 < \alpha_2 < \alpha_0$. Following a similar argument to that in (29) there exists $\varepsilon_1 > 0$ such that, for any $0 < \mu < \varepsilon_1$,

$$\begin{cases} x_i(\mu) \leq \frac{\mu}{\alpha_2}, & i \in I^0, \\ z_i(\mu) \leq \frac{\mu}{\alpha_2}, & i \in \{1, \dots, n\} \setminus I^0, \end{cases}$$

and, so,

$$(x(\mu), y(\mu), z(\mu)) \in \mathcal{F}^0\left(\frac{\mu}{\alpha_2}e\right), \text{ whenever } 0 < \mu < \varepsilon_1.$$

On the other hand, since $\frac{\alpha_2}{\alpha_1} > 1$ and $\text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0)) > 0$ (as stated in (i)), then $\frac{\alpha_2}{\alpha_1} \text{clm}\mathcal{F}^0(0_n, (x^0, y^0, z^0))$ is a particular calmness constant for

\mathcal{F}^0 at $(0_n, (x^0, y^0, z^0))$, which guarantees the existence of $0 < \varepsilon < \varepsilon_1$ such that

$$\begin{aligned} d((x(\mu), y(\mu), z(\mu)), \mathcal{F}^0(0_n)) &\leq \frac{\alpha_2}{\alpha_1} \text{clm} \mathcal{F}^0(0_n, (x^0, y^0, z^0)) d\left(\frac{\mu}{\alpha_2} e, 0_n\right) \\ &= \kappa_1 \mu, \text{ whenever } 0 < \mu < \varepsilon. \end{aligned}$$

Finally, the desired inequality comes from the easily checkable fact that $\mathcal{F}^0(0_n) \subset \Lambda$. ■

5 Conclusions

The main part of the present paper, and where the main technical difficulties appear, is devoted to the derivation of a formula for the calmness modulus of the feasible set mapping, \mathcal{F} , associated with a linear inequality systems including an abstract constraint (see (2)); in other words, \mathcal{F} is the feasible set mapping associated with a partially perturbed system (where some constraints remain unchanged). It is clear from the definitions that this calmness modulus is always smaller than or equal to the modulus associated with perturbations of all constraints. In this sense, the expression given in Theorem 4 constitutes a refinement of previous results which can be traced from [9, Sections 4 and 5]. Moreover, we point out the fact that the arguments behind this refinement are notably different from the ones of [9].

The expression of $\text{clm} \mathcal{F}((\bar{a}, \bar{b}), \bar{x})$ given in Theorem 4 (where the abstract constraint set, C , is polyhedral) is *point-based* in the sense that it only depends on the nominal data. In view of Theorem 4, we are providing a conceptually implementable procedure for computing this calmness modulus since it involves the computation of the distance from the origin to a finite number of polyhedra only depending on the nominal data.

As commented above, Section 4 develops an illustration related to the classical central path construction. The main result in this direction is Theorem 6, which provides an expression for the linear rate of convergence of the central path in terms of the calmness modulus of the feasible set mapping \mathcal{F}^0 defined in (25) and the scalar α_0 defined in (27). This theorem establishes the existence of $\varepsilon > 0$ such that

$$d((x(\mu), y(\mu), z(\mu)), \Lambda) \leq \kappa \mu,$$

whenever $0 < \mu < \varepsilon$, and $\kappa > \alpha_0^{-1} \text{clm} \mathcal{F}^0(0_n, (x^0, y^0, z^0))$.

For comparative purposes, we comment that the classical results of [17, Theorem 2.1 and Corollary] establish the existence of $\varepsilon > 0$ such that

$$\|x(\mu) - x^0\| \leq \kappa\mu \text{ and } \|z(\mu) - z^0\| \leq \kappa\mu, \quad (32)$$

whenever $0 < \mu < \varepsilon$ and $\kappa > \max\{\|x'(0)\|, \|z'(0)\|\}$, where $x'(0)$ and $z'(0)$ are the derivatives of $x(\cdot)$ and $z(\cdot)$ at 0^+ .

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